Rob Zima ‘08: “Non-negativity of the \( \gamma \)-vector for 3-dimensional polytopes”

In this honors project, Rob Zima looked at patterns in the angles of polyhedra. In math, the search for patterns helps us to better understand the world around us. We look to extend patterns, asking questions about which circumstances guarantee a certain outcome.

*How do you measure the angles of a polyhedron?*

We are used to measuring angles in the plane – 2-dimensional angles. However we measure the angles, we are comparing to a “whole-circle” angle. An angle of 90° is \( \frac{1}{4} \) of the total 360° or an angle of \( \pi \) radians is \( \frac{1}{2} \) of the total 2\( \pi \) radians. In either case, we can think of the angle as the fraction of directions we can move from the vertex to the interior of the angle. We do the same thing to measure angles in other dimensions: the angle at any face of the polyhedron is the fraction of the directions we can move from that face and go inside the shape. When we are on a face of the polyhedron, half the directions go in and half go out, so we say the angle at a face is \( \frac{1}{3} \). At the vertex of a cube, the angle is \( \frac{1}{8} \) because that is the fraction of directions that go into the cube (see diagram). For most shapes, these angles are very hard to compute exactly.

We define angle sums as the sum of all the faces in a certain dimension: \( \alpha_0 \) is the sum of all the angles at 0-dimensional faces or vertices; \( \alpha_1 \) is the sum of all the angles at 1-dimensional faces or edges; \( \alpha_2 \) is the sum of all the angles at 2-dimensional faces, our regular faces. For a cube, there are 8 vertices that each have angle \( \frac{1}{8} \), so \( \alpha_0 = \frac{1}{8} \times 8 = 1 \). The angle at an edge is \( \frac{1}{4} \) and there are 12 edges, so \( \alpha_1 = \frac{1}{4} \times 12 = 3 \). Similarly, \( \alpha_2 = \frac{1}{2} \times 6 = 3 \). If you imagine a higher dimensional polyhedron, you get some sense of what we call a polytope and the angles can be defined the same way there.
What sorts of patterns can you find in the angles of polyhedra?

There is a classical pattern called Gram’s equation, which tells us that the angle sum at the vertices plus the angles at the faces equals the angle sum at the edges plus 1: \( \alpha_0 + \alpha_2 = \alpha_1 + 1 \), which can be rewritten as \( \alpha_0 - \alpha_1 + \alpha_2 - 1 = 0 \). For example, with the cube above, 1-3-3-1=0. This pattern generalizes to higher dimensional polytopes, where adding and subtracting the angles at different dimensions equals 0: \( \sum_{i=0}^{d} \alpha_i = 0 \), where the polytope is dimension \( d \) and we define \( \alpha_d = 1 \). We call a certain way of adding and subtracting these angle sums the \( \gamma \)-vector and since we are subtracting as well as adding, there is no guarantee that all the entries are positive. However, Dr. Kristin Camenga has proved that the \( \gamma \)-vector of any tetrahedron or 4-dimensional tetrahedron are non-negative. In his honors project, Rob Zima proved that all pyramids have non-negative \( \gamma \)-vector and he also put tighter bounds on \( \gamma \)-vectors of prisms.

What other questions can you ask about angles in polyhedra?

Rob answered the question about whether the \( \gamma \)-vector is non-negative for pyramids. But we still don’t know whether the \( \gamma \)-vector is non-negative for prisms! And is it non-negative for ALL polyhedra in 3-dimensions? Even if that is true, what about four dimensions? Or five? And what about other patterns in the \( \gamma \)-vector? Sometimes the \( \gamma \)-vector is non-decreasing: each number is bigger than the last! For which polyhedra will this be true? Maybe you can think up another question, too!

In math, answering questions frequently leads to asking more questions. That’s why we never run out of questions!

About Rob Zima

Rob writes: “I came to Houghton College from Buffalo, NY and received a Bachelor’s Degree in Mathematics/Adolescence Education.

My love of this topic was stated so explicitly by Galileo who wrote, "Mathematics is the language in which God wrote the universe." There will always be questions that need answers, problems that need solutions, and conjectures that need proofs. In a sense, I worship my Creator every day I go to work.

This honors project provided me an opportunity to become engaged in current problems that have no solutions; to think outside the box, develop new ideas and research habits that have guided me through graduate school.

I’m currently pursuing a Masters degree in Mathematics at the University of Kansas in Lawrence, KS. On the side, I particularly enjoy watching college basketball (Go JayHawks!) and serving at my church throughout the week.”