Problem 2: Assume that a system is comprised of two particles that interact only gravitationally. Define \( \hat{e} \) as a unit vector in the direction of \( \hat{e} \). Define \( \hat{p} \) as a unit vector 90° from \( \hat{e} \) such that \( \hat{e} \times \hat{p} = \hat{h} \) (where \( \hat{h} \) is a unit vector parallel to orbital angular momentum).

(a) Express the position vector in terms of \( r, \Theta^*, \hat{e}, \) and \( \hat{p} \). Demonstrate that the velocity can be written:

\[
\hat{v} = \frac{\mu}{r(1+e)} \left[ -\sin \Theta^* \hat{e} + (e + \cos \Theta^*) \hat{p} \right]
\]

(b) At the point of intersection of an elliptical orbit with the semi-minor axis, prove that the following relationships are true:

\[
\begin{align*}
r &= a \\
v &= \frac{\mu}{r} \\
\Theta^* &= \cos^{-1}(-e)
\end{align*}
\]

c) Start with the conic equation and differentiate it. Prove that \( r \) possesses a maximum magnitude at the ends of the semi-latus rectum; determine that the corresponding value is \( \pm e \sqrt{\mu/p} \).

2a) First, a diagram is drawn to label the appropriate quantities for the problem.
Based on the way $\theta^*$ is defined, in terms of the $\hat{e}$ and $\hat{p}$ unit vectors, the position vector of the second body, $\vec{r}^b$, is given by:

$$\vec{r}^b = r \cos \theta^* \hat{e} + r \sin \theta^* \hat{p}$$

Notice that this is the position of the orbiting body relative to the orbited body. We therefore can find the relative velocity of the orbiting body with respect to the orbited body by differentiating:

$$\vec{v}^b = \vec{v} = [r \cos \theta^* - r \hat{e} \sin \theta^*] \hat{e} + [r \sin \theta^* + r \hat{p} \cos \theta^*] \hat{p}$$

Recall that the value of $r$ in terms of the semi-latus rectum, $p$, the eccentricity $e$, and the angle $\theta^*$ is:

$$r = \frac{p}{1 + e \cos \theta^*} = \frac{p}{1 + e \cos \theta^*}$$

Differentiating we find:

$$\dot{r} = \frac{-p (1 + e \cos \theta^*)^2 (-e \dot{\theta}^* \sin \theta^*)}{(1 + e \cos \theta^*)^2}$$

$$\dot{r} = \frac{pe \dot{\theta}^* \sin \theta^*}{(1 + e \cos \theta^*)^2} = \frac{r^2 \dot{\theta}^* \sin \theta^*}{p}$$

Recall the specific angular momentum, $h$:

$$h = r^2 \dot{\theta}^* = \sqrt{\mu p}$$

where $\mu = G (m_1 + m_2)$ where $m_1$ and $m_2$ are the masses of the 2 bodies.