8.710 Problem Set #5 – #4

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Suppose that δ_0 and δ_1 are the phase shifts at an energy E for scattering $l=0$ and $l=1$ neutrons respectively from a nucleus.

(a) Derive the differential cross section $d\sigma/d\Omega$ in terms of δ_0 and δ_1 displaying its explicit dependence on θ, the scattering angle.

Solution: Consider first the incident wave:

$$\psi_0 = \left(\frac{1}{2\pi}\right)^{3/2} e^{ikz} = \left(\frac{1}{2\pi}\right)^{3/2} e^{ikr\cos\theta}$$

This can be written in terms of Legendre polynomials:

$$\psi_0 = \left(\frac{1}{2\pi}\right)^{3/2} \sum_{l=0}^{\infty} (2l+1) P_l(\cos\theta)\, i^l\, j_l(kr)$$

Now, lets consider the actual Schrödinger equation in the presence of the potential:

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2} + K^2\right]\psi = 0$$

where $K^2 = \frac{2\mu}{\hbar^2}(E-V)$.

The solutions to this equation are spherical Bessel functions. We have both types of

Bessel functions: $j_\ell(kr)$ and $n_\ell(kr)$. Only the $j_\ell(kr)$ solution is well behaved at the origin. The spherical Bessel functions have the following large r behavior:

$$j_\ell(kr) \xrightarrow{r\to\infty} \frac{1}{kr}\sin\left(kr - \frac{\ell\pi}{2}\right).$$

Thus, as $r \to \infty$:

$$\psi_0 = \frac{1}{kr}\sum_{\ell=0}^{\infty}(2\ell+1)P_\ell(\cos\theta)\,i^\ell\sin\left(kr-\frac{\ell\pi}{2}\right)$$

$$\psi_0 = \frac{1}{kr}\sum_{\ell=0}^{\infty}(2\ell+1)P_\ell(\cos\theta)\frac{i^{\ell-1}}{2}$$

$$\Big\{\underbrace{e^{i(kr-\frac{\ell\pi}{2})}}_{\text{outgoing wave}} - \underbrace{e^{-i(kr-\frac{\ell\pi}{2})}}_{\text{incoming wave}}\Big\}$$

Clearly, for our final ψ, we will have a wave with the same incoming wave part, since this part will have not yet seen the potential. The outgoing wave however will be changed, by an amount we will call $2i\delta_\ell$:

$$\psi = \frac{1}{kr}\sum_{\ell=0}^{\infty}(2\ell+1)P_\ell(\cos\theta)\,\frac{i^{\ell-1}}{2}$$

$$\left\{e^{i(kr-\frac{\ell\pi}{2}+2\delta_\ell)} - e^{-i(kr-\frac{\ell\pi}{2})}\right\}.$$

Now, we know that by definition:

$$\psi = \psi_0 + \psi_{sc}$$

$$\therefore \psi_{sc} = \psi - \psi_0$$

$$\psi_{sc} = \frac{1}{kr}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)\frac{i^{l-1}}{2}\Big\{e^{i(kr-\frac{l\pi}{2}+2\delta_l)} - \cancel{e^{-i(kr-\frac{l\pi}{2})}} - e^{i(kr-l\pi/2)} + \cancel{e^{-i(kr-\frac{l\pi}{2})}}\Big\}$$

$$\psi_{sc} = \frac{1}{kr}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)\frac{i^{l-1}}{2}e^{i(kr-\frac{l\pi}{2})}(1-e^{2i\delta_l})$$

$$\psi_{sc} = \frac{e^{ikr}}{r}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)\,i^l\,\cancel{e^{-i\frac{l\pi}{2}}}^{(-i)^l}\frac{1}{k}\sin\delta_l\,e^{i\delta_l}$$

$$\boxed{\psi_{sc} = \frac{e^{ikr}}{r}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)\frac{1}{k}\sin\delta_l\,e^{i\delta_l}}$$

We also have our standard definition of the scattering amplitude, however, which is:

$$\psi_{sc} = \frac{e^{ikr}}{r} f(\theta) .$$

By comparison then we see:

$$\boxed{f(\theta) = \sum_{l=0}^{\infty} (2l+1) P_l(\cos\theta) \frac{1}{k} \sin\delta_l \, e^{i\delta_l}}$$

The differential cross section then is:

$$\frac{d\sigma}{d\Omega} = \left| f(\theta) \right|^2$$

$$\frac{d\sigma}{d\Omega} = \left| \sum_{l=0}^{\infty} (2l+1) P_l(\cos\theta) \frac{1}{k} \sin\delta_l \, e^{-i\delta_l} \right|^2$$

If we neglect all δ_l but δ_1 and δ_0,

$$\frac{d\sigma}{d\Omega} = \Big| P_0(\cos\theta) \frac{1}{k} \sin\delta_0 \, e^{-i\delta_0} + 3 P_1(\cos\theta) \frac{1}{k} \sin\delta_1 \, e^{-i\delta_1} \Big|^2$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{k^2} \left| \sin\delta_0 \, e^{-i\delta_0} + 3 \cos\theta \sin\delta_1 \, e^{-i\delta_1} \right|^2$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{k^2}\left(\sin\delta_0\, e^{-i\delta_0} + 3\cos\theta \sin\delta_1\, e^{-i\delta_1}\right)$$
$$\left(\sin\delta_0\, e^{i\delta_0} + 3\cos\theta\sin\delta_1\, e^{i\delta_1}\right)$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{k^2}\left\{\sin^2\delta_0 + 9\cos^2\theta\sin^2\delta_1 + 3\sin\delta_0\sin\delta_1\cos\theta\left(e^{i(\delta_1-\delta_0)} + e^{-i(\delta_1-\delta_0)}\right)\right\}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{k^2}\left\{\sin^2\delta_0 + 9\cos^2\theta\sin^2\delta_1 + 6\sin\delta_0\sin\delta_1\cos\theta\cos(\delta_1-\delta_0)\right\}$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{k^2}\left\{\sin^2\delta_0 + 9\cos^2\theta\sin^2\delta_1 + 6\sin\delta_0\sin\delta_1\cos\theta\cos(\delta_1-\delta_0)\right\}}$$

(b) Derive the total cross section

$$\sigma = \int \frac{d\sigma}{d\Omega}\, d\Omega.$$

Notice that the interference part present in $d\sigma/d\Omega$ vanishes in σ.

<u>Solution</u>: We are told that:

$$\sigma = \int \frac{d\sigma}{d\Omega}\, d\Omega$$

$$\sigma = 2\pi \int_0^{\pi} \sin\theta\, d\theta\, \frac{1}{k^2} \left\{ \sin^2\delta_0 + 9\cos^2\theta \sin^2\delta_1 + 6\sin\delta_0 \sin\delta_1 \cos\theta \cos(\delta_1 - \delta_0) \right\}$$

$$\sigma = \frac{2\pi}{k^2} \left\{ -\sin^2\delta_0 \cos\theta \Big|_0^{\pi} + 9\sin^2\delta_1\, \frac{1}{3}\cos^3\theta \Big|_0^{\pi} + 6\sin\delta_0 \sin\delta_1 \cos(\delta_1 - \delta_0)\, \frac{1}{2}\cancelto{0}{\cos^2\theta} \Big|_0^{\pi} \right\}$$

$$\boxed{\sigma = \frac{2\pi}{k^2} \left\{ 2\sin^2\delta_0 + 6\sin^2\delta_1 \right\}}$$

c) Sketch $d\sigma/d\Omega$ for the following pairs:

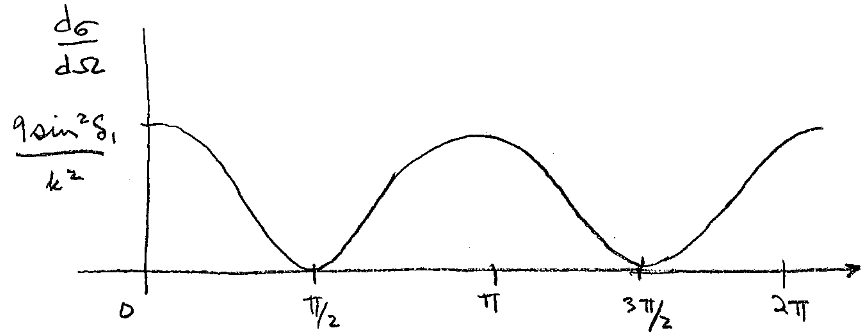
$\delta_0 = 0$	$\delta_0 = \delta_0$	$\delta_0 = 0$	$\delta_0 = \pi/2$
$\delta_1 = \delta_1$	$\delta_1 = 0$	$\delta_1 = \pi/2$	$\delta_1 = \pi/2$.

<u>Solution</u>:

a) $\delta_0 = 0 \quad \delta_1 = \delta_1$

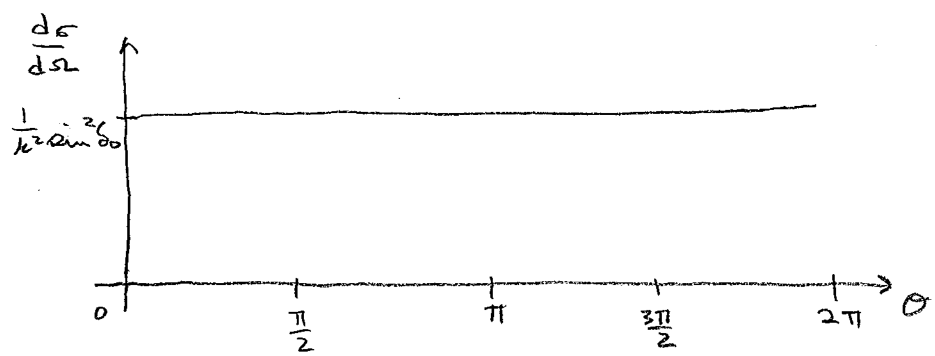
$$\therefore \quad \frac{d\sigma}{d\Omega} = \frac{1}{k^2} \left\{ 9\cos^2\theta \sin^2\delta_1 \right\}$$

Hence the plot:



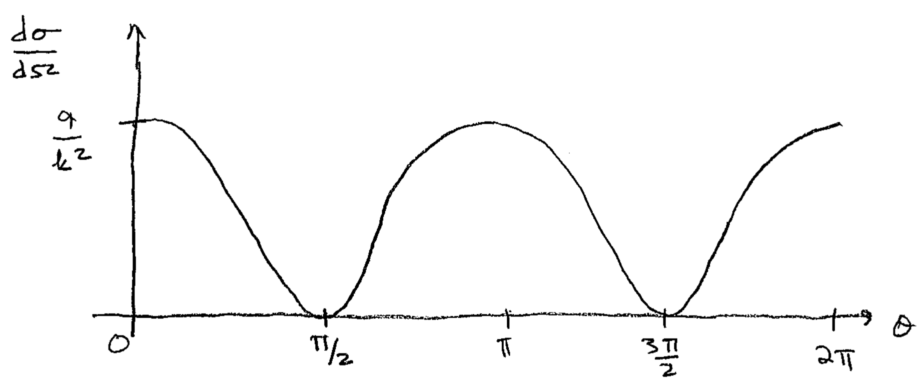
b) $\delta_0 = \delta_0 \qquad \delta_1 = 0$

$$\therefore \quad \frac{d\sigma}{d\Omega} = \frac{1}{k^2}\left\{ \sin^2 \delta_0 \right\}$$



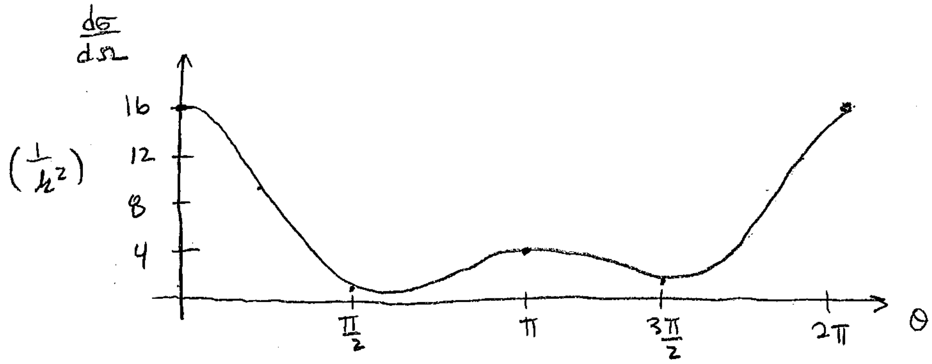
c) $\delta_0 = 0 \qquad \delta_1 = \pi/2$

$$\therefore \quad \frac{d\sigma}{d\Omega} = \frac{1}{k^2}\left\{ 9\cos^2\theta \right\}$$



d) $\delta_0 = \frac{\pi}{2}$ $\delta_1 = \frac{\pi}{2}$

$$\therefore \frac{d\sigma}{d\Omega} = \frac{1}{k^2}\left\{1 + 9\cos^2\theta + 6\cos\theta\right\}$$



θ	$1 + 9\cos^2\theta + 6\cos\theta$
0°	16
45°	9.7
90°	1
135°	1.3
180°	4
225°	1.3
270°	1
315°	9.7